PHYS-100C, Lectures 9-11 (WAVEGUIDES)

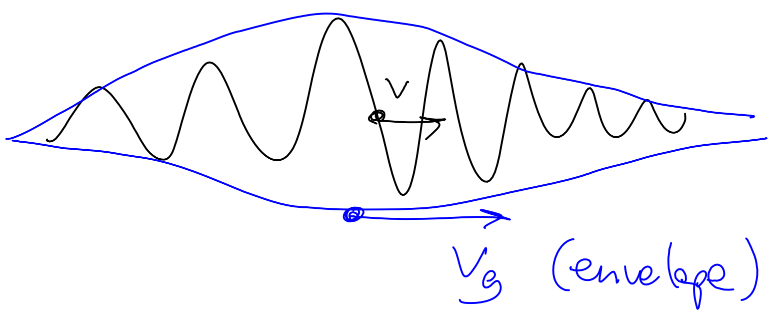
Monday, April 19, 2010
6:30 AM

* Dispersion:

ε ($\Rightarrow n, v$) depends on ω

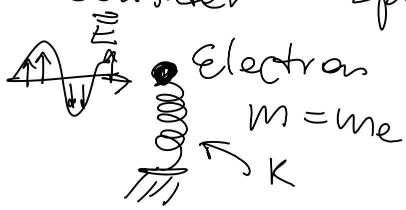
Phase velocity $v = \dfrac{\omega}{k}$

Group velocity $v_g = \dfrac{\partial \omega}{\partial k}$



Why does ε (and v, n etc.) depend on ω?

Consider "spring" model of electron:



$$\omega = \sqrt{\frac{k}{m_e}}$$

Eq. of motion:

$$\underbrace{m\frac{\partial^2 x}{\partial t^2}}_{F=ma} + \underbrace{m\gamma\frac{\partial x}{\partial t}}_{\text{dissipation}} + \underbrace{m\omega_0^2 x}_{kx} = \underbrace{eE_0 \cdot \cos\omega t}_{\text{driving force}}$$

General Solution: $x = \tilde{x}_0 \cdot e^{-i\omega t}$

$$-\omega^2 x - i\gamma x + \omega_0^2 x = \frac{eE_0}{m} e^{-i\omega t}$$

$$\tilde{x}_0 = \frac{e/m}{\omega_0^2 - \omega^2 - i\gamma} E_0$$

Polarization $\tilde{p} = e\tilde{x}$

$$\tilde{P} = \sum_{\substack{\text{ALL} \\ \text{ELECTRONS/VOLUME}}} \tilde{p} = \varepsilon_0 \tilde{\chi}_e \tilde{E}$$

$$\tilde{\varepsilon} = \varepsilon_0 (1 + \tilde{\chi}_e)$$

$$\tilde{\varepsilon}_R = 1 + \frac{Ne^2}{m\varepsilon_0} \cdot \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma\omega}$$

Planar wave $E = E_0 \cdot e^{i(\tilde{k}z - \omega t)}$

where $\tilde{k} = \sqrt{\tilde{\varepsilon}\mu_0}\, \omega = k + i\frac{\alpha}{2}$

$$E = E_0 \cdot e^{-\frac{\alpha z}{2}} \cdot e^{i(kz - \omega t)}$$

α is absorption coefficient

$$n = \frac{ck}{\omega}$$

for small dispersive corrections:

$$\sqrt{1+\varepsilon} \simeq 1 + \varepsilon/2$$

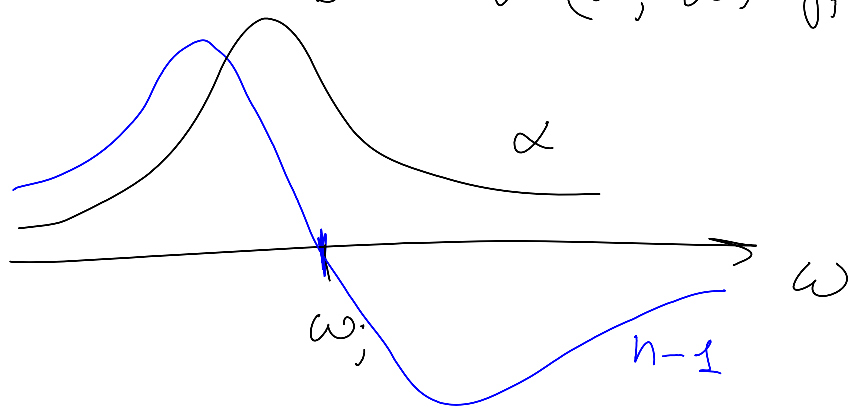
$\tilde{k} \ldots \omega \; [\; 1 \;\; Ne^2 \sum \;\; f_j \;\;]$

$$\tilde{k} = \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$$

and

$$\alpha \cong \frac{Ne^2\omega^2}{m\varepsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$$



Far away from resonances,

$$n \sim 1 + \ldots \sum \frac{\ldots}{\omega_j^2 - \omega^2}$$

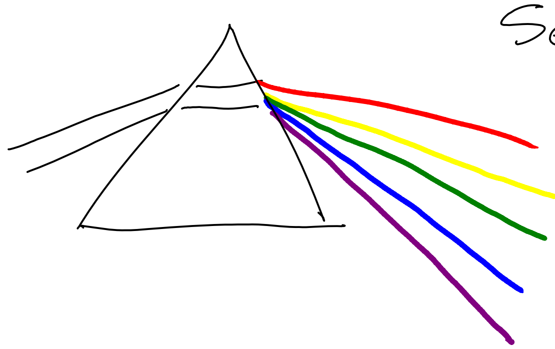
ω_j are in UV range

for visible light $\omega < \omega_j$

$$\frac{1}{\omega_j^2-\omega^2} = \frac{1}{\omega_j^2(1-\omega^2/\omega_j^2)} \simeq \frac{1}{\omega_j^2}\left(1+\frac{\omega^2}{\omega_j^2}\right)$$

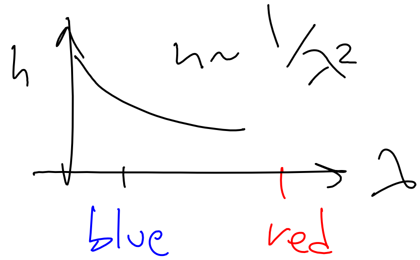
$$n = 1 + A\left(1 + B/\lambda^2\right)$$ Cauchy's FORMULA

* Physics consequences:

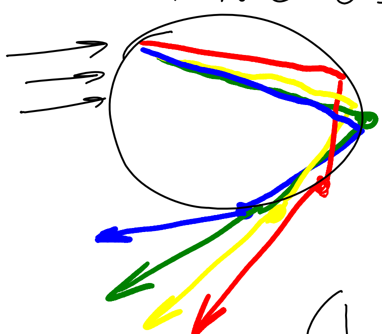


See cover art "Dark side of the Moon" by PINK FLOYD

n increases for shorter wavelengths (blue/violet)



RAINBOWS:

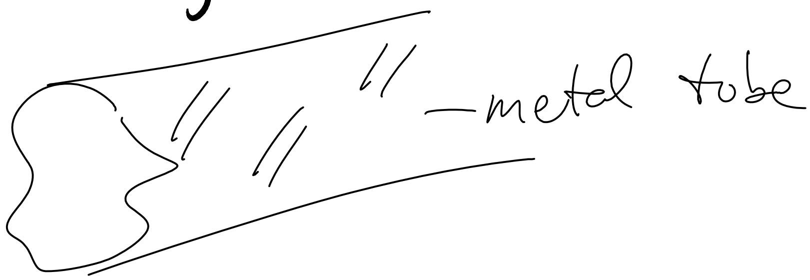


UV light (bad for skin, can damage DNA) gets substantially absorbed by atmosphere (since $\omega_j \approx$ UV)

For very large frequencies

$\omega > \omega_j \qquad n < 1$ (X-RAYS)

* Waveguides



Perfect Conductor:

$E = 0$, $B = 0$ inside

Boundary conditions:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma \qquad \text{(i)}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \text{(ii)}$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad \text{(iii)}$$

$$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = K_f \times n \qquad \text{(iv)}$$

OR $E^{\parallel} = 0$, $B^{\perp} = 0$ (ii), (iii)

(since $E = 0$, $B = 0$ inside)

In interior of the waveguide

$$\bar{\nabla} \cdot \bar{E} = 0 \qquad \bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B} = 0 \qquad \nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t}$$

No free charges, currents, assume vacuum (thus $1/c^2$)

$$\tilde{E} = \tilde{E}_0 \cdot e^{i(kz - \omega t)}$$

$$\tilde{B} = \tilde{B}_0 \cdot e^{i(kz - \omega t)}$$

E, B are not generally transverse (which was a result of plane waves with no x-y dependence for E, B)

$$\nabla\times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

For (iii):

$$(z): \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \quad (1)$$

$$(x): \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x \quad (2)$$

$$(y): ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y \quad (3)$$

$$\left(\text{Since } \frac{\partial E_y}{\partial z} = ikE_y \quad \frac{\partial E_x}{\partial z} = ikE_x\right)$$

Similarly for Eq.(iv):

$$(z): \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \quad (4)$$

$$(x): \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \quad (5)$$

$$(y): ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \quad (6)$$

From (3) & (5), eliminate B_y:

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

From (2) & (6), eliminate B_x

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

Similarly for B_x, B_y:

(2) and (6):

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

E_z & B_z determine everything else.

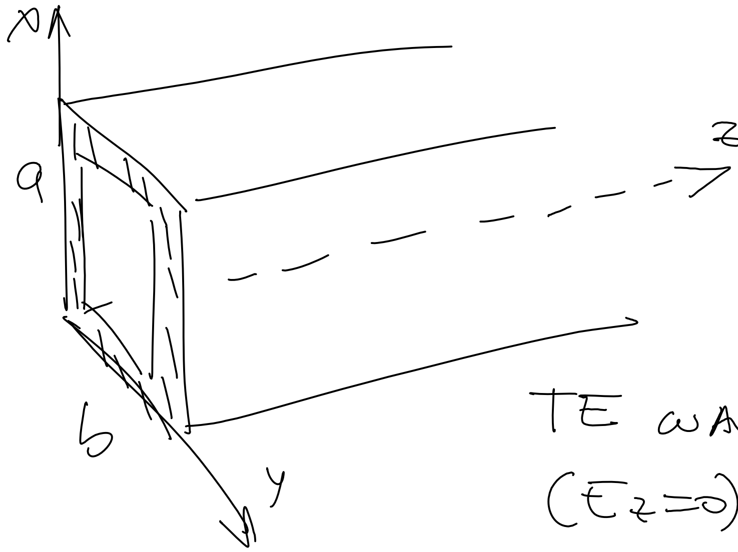
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right)\begin{matrix}E_z,\\ B_z\end{matrix} = 0$$

$E_z = 0 \Rightarrow$ TE waves

$B_z = 0 \Rightarrow$ TM waves

$E_z = B_z = 0 \Rightarrow$ TEM waves (cannot occur in hollow WG)

* Rectangular Waveguide:



TE waves

$(E_z = 0)$

$$E_z,\ B_z(x,y) = X(x)\,Y(y)$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]XY = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

$$k_x^2 + k_y^2 = \left(\frac{\omega}{c}\right)^2 - k^2$$

$$X = A\sin(k_x \cdot x) + B\cdot\cos(k_x \cdot x)$$

at $x=0$; $x=a$ $\quad B_x = 0$

$$B_x = \ldots \underbrace{\frac{\partial B_z}{\partial x}}_{\frac{\partial X}{\partial x}\cdot Y = 0} - \ldots \underbrace{\frac{\partial E_z}{\partial y}}_{\to\, 0\ (\mathrm{TE})} = 0$$

$$\frac{\partial X}{\partial x} = k_x \left(A \cdot \cos(k_x \cdot x) - B \sin(k_x \cdot x)\right) = 0$$

for $x = 0, a \Rightarrow A = 0$

$$\sin(k_x \cdot x) = 0 \quad \text{at } x = a$$

$$k_x = \frac{m\pi}{a} \qquad m = 0, 1, 2 \ldots$$

Similarly $k_y = \frac{n\pi}{b} \quad n = 0, 1, 2, \ldots$

$$B_z = XY = B_0 \cdot \cos\frac{m\pi x}{a} \cdot \cos\frac{n\pi x}{b}$$

TE_{mn} mode

Standing waves (2D) in xy

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi n}{b}\right)^2}$$

$$\omega_{mn}^2 = \left(\frac{\pi m c}{a}\right)^2 + \left(\frac{\pi n c}{b}\right)^2$$

cut-off frequency for TE_{mn}

If $\omega < \omega_{mn}$, k is complex (attenuation)

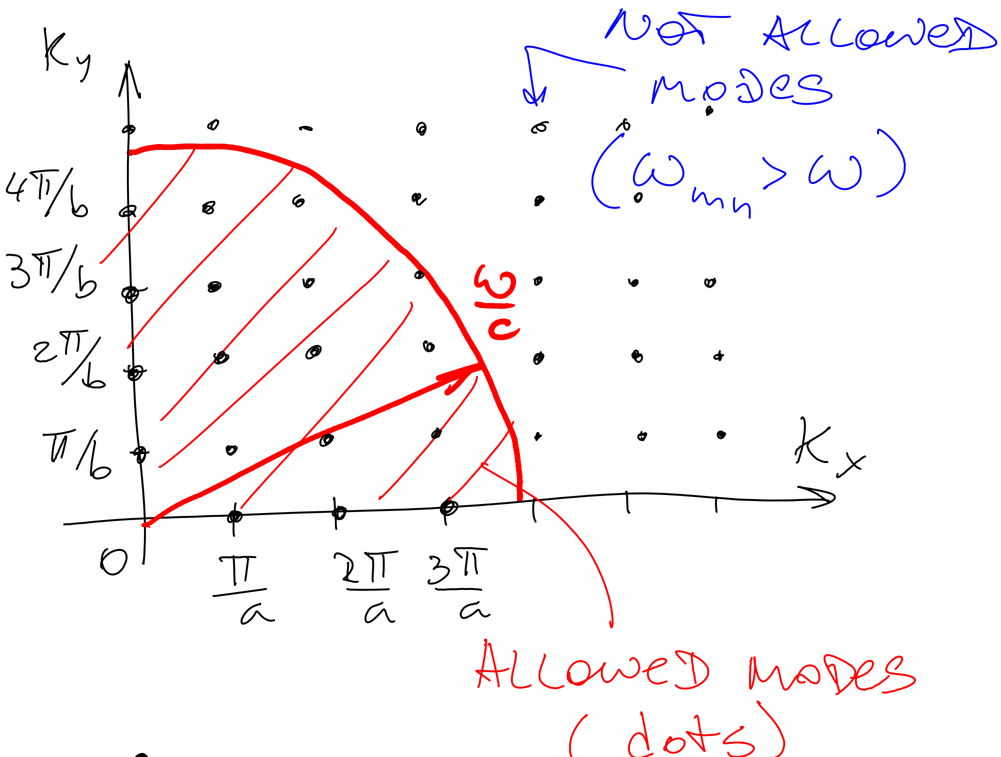
$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} \quad (> c)$$

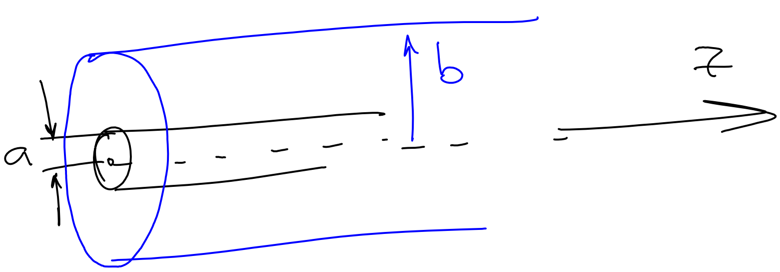
Group velocity:

Group velocity:

$$V_g = \frac{\partial \omega}{\partial k} = \frac{1}{\partial k / \partial \omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$



* Coaxial cable:



SOLUTION ALLOWS TEM
$(B_z = E_z = 0)$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z = 0$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z = 0$$

$$\nabla \cdot E = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

$$\nabla \cdot B = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Electro/Magneto-statics:

$$\vec{E} = \frac{A \cdot \cos(kz - \omega t)}{r} \cdot \hat{r}$$

$$\vec{B} = \frac{A \cdot \cos(kz - \omega t)}{cr} \cdot \hat{\phi}$$